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# INSTABILITIES IN THIN ELASTIC–PLASTIC TUBES

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**Abstract**—The condition for instability of an isotropic elastic–plastic cylinder under pressure $p$ and independent axial load $P$ is examined. It is shown that the boundary value problem admits different loading conditions, each condition leading to a different result. The critical stress for axially loaded cylinders and for cylinders under uniform lateral pressure follow as special cases. It is shown, further, that the value of the critical stress is quite sensitive to the variation in the normal to the yield surface at the local stress point.

## NOTATIONS

| | |
|---|---|
| $\chi, \theta, r$ | cylindrical coordinate |
| $L, a, t$ | Length, mean radius and thickness of the cylinder |
| $\zeta$ | distance from the middle surface |
| $p$ | pressure |
| $P$ | independent axial load |
| $P_z$ | total axial load |
| $\sigma_{ij}, s_{ij}, \tau_{ij}$ | components of the true stress, the nominal stress and the Kirchhoff stress |
| $\sigma_1$ | axial stress |
| $\sigma_2$ | hoop stress |
| $\alpha$ | stress ratio $(\sigma_2/\sigma_1)$ |
| $T_i, \dot{T}_i$ | boundary traction and traction-rate |
| $(\dot{\ })$ | the material derivative |
| $n_i$ | normal to the boundary surface |
| $v_i$ | components of particle velocity |
| $u, v, w$ | velocities at the middle surface |
| $\dfrac{D\tau_{ij}}{Dt}$ | the Jaumann derivative of the Kirchhoff stress |
| $\varepsilon_{ij}$ | rate of deformation tensor |
| $\omega_{ij}$ | spin tensor |
| $2\mu$ | rigidity modulus |
| $E$ | Young's modulus |
| $E_t$ | tangent modulus |
| $\nu$ | Poisson's ratio |
| $h$ | work hardening parameter |
| $m_{ij}$ | normal to the yield surface |
| $m_1, m_2, m_3$ | components of $m_{ij}$ in principal stress space |
| $m, n$ | positive integers |

## INTRODUCTION

THE INSTABILITY of a closed tube under internal pressure and independent axial load was investigated by Swift [1], Marciniak [2], Lankford and Saibel [3], and Hillier [5]. Each of the above authors obtained different values of the critical stress at instability. Mellor [6] sought to correlate the difference in the predicted values of the critical stress experimentally and found that under proportional stressing the theory proposed by Lankford and Saibel and by Mellor agrees well with experiment. In the papers cited, the cylinder was assumed to be long, so that the end effect could be neglected, and made of rigid plastic solid obeying von Mises yield criterion.

In the current investigation, an elastic–plastic cylindrical shell under continuing deformation is considered. The critical condition for stability of the cylinder under internal pressure $p$ and independent axial load $P$ is obtained by Hill's variational technique [7], modified to include the effect of the fluid pressure. In the formulation, the plastic component of the deformation is expressed in terms of the normal to the yield surface at the local stress point. Hence, the analysis can be applied to a wide class of inelastic solids.

The theoretical results of the above-cited authors are obtained as special cases from the critical condition for stability for a rigid–plastic von Mises solid. Differences in their results are due to different boundary conditions imposed by each author. This observation sets aside any controversy regarding the validity of any particular theory.

The result of the present investigation is also applicable for materials for which the elastic-component of the deformation is of the same order of magnitude as the plastic component and hence can no longer be neglected. The result obtained here is expected to be useful for application to pressure vessels where the above condition may often hold.

Yet another feature of the present analysis is the dependence of the critical stress on the unit normal to the local yield surface. Thus, any variation in the shape of the yield surface at the local stress point is reflected directly on the value of the critical stress at instability. The variation in the shape may be due to departure from the assumed loading condition or due to imperfect material behaviour.

The sensitivity of the critical stress to variation in the shape of the yield surface has been investigated by Sewell [11] for a plate under in-plane external load, and by Ariaratnam and Dubey [12] for a cylindrical shell under axial compressive load. However, the effect on the critical stress of variation in the shape of the yield surface for an elastic–plastic cylindrical shell under internal pressure $p$ and independent axial load $P$ seems not to have been investigated before. In this connection, it may be mentioned that Hillier [13] found that the critical subtangent at instability for a von Mises solid differs significantly from that for a Tresca solid. However, he considered only the rigid–plastic behaviour for the material.

## STATEMENT OF THE PROBLEM

Consider an elastic–plastic cylinder closed at both ends, and subjected to internal pressure $p$ and independent axial load $P$. At the instant under consideration, the mean radius $a$, the thickness $t$ ($t \ll a$) and the length $L$ of the cylinder is supposed given. The length of the cylinder is assumed to be large compared to the radius so that the end effect may be neglected. We consider the configuration of the cylinder in the current state to be the reference configuration, and take a cylindrical coordinate system, origin at the centre of the axis, as reference frame. In the reference configuration, the components of the true stress $\sigma_{ij}$, the nominal stress $s_{ij}$ and the Kirchhoff stress $\tau_{ij}$ are identical and its only non-zero components are $\sigma_1$ and $\sigma_2$, where

$$P_z = P + p(\pi r^2) = \sigma_1(2\pi r t), \quad \text{and} \quad pr = \sigma_2 t. \tag{1}$$

The current stress distribution is assumed uniform.

The stability of the cylinder is now tested by applying a virtual velocity field $v_i$. During the incremental displacement, the boundary conditions are prescribed in terms of the nominal stress and the material derivative, that is the time derivative following the element, of the nominal stress.

$$T^j = n_i s^{ij}, \tag{2a}$$

$$\dot{T}^j = n_i \dot{s}^{ij}. \tag{2b}$$

Using the relation (Hill [7])

$$\dot{s}^{ij} = \dot{\sigma}^{ij} + \sigma^{ij} v^k{}_{,k} - \sigma^{jk} v^i{}_{,k} \tag{3}$$

the boundary condition (2b) can be expressed in terms of the material derivative of the true stress $\sigma_{ij}$,

$$\dot{T}^j = n_i(\dot{\sigma}^{ij} + \sigma^{ij} v^k_{,k} - \sigma^{jk} v^i_{,k}). \tag{4}$$

Here and subsequently, the comma followed by a subscript denotes covariant differentiation with respect to that subscript.

During the virtual displacement, the internal work minus external work is easily shown to be

$$\delta K = \tfrac{1}{2}(\delta t)^2 \left[ \int_V \dot{s}^{ij} v_{j,i}\, \mathrm{d}V - \int_S \dot{T}^j v_j\, \mathrm{d}S \right]$$

where $(\delta t)$ is the time during which the displacement is $v_i \delta t$. The volume and the surface integrations are to be carried over the initial volume and the initial surface area, respectively.

If $\delta K$ is positive for any $v_i$, the cylinder is certainly stable, as more energy is required by the cylinder to achieve corresponding displacement than can be supplied by the external work. For an actual field, $\delta K$ vanishes. For $\delta K < 0$, only a part of the external work is taken up by the internal deformation; the difference is still available to cause further deformation which results in further increase in $\delta K$. This state of affairs, if continued, leads to ultimate failure. For convenience, we write

$$\delta E = \int_V \dot{s}^{ij} v_{j,i}\, \mathrm{d}V, \tag{5a}$$

$$\delta W = \int_S n_i \dot{s}^{ij} v_j\, \mathrm{d}S. \tag{5b}$$

Hence, for stability,

$$\delta E - \delta W > 0. \tag{6}$$

For dead loading on the part $S_F$ of the boundary surface $S$, and rigid constraints on the remainder $S - S_F$, $\delta W = 0$ and condition (6) reduces to the stability criterion (6) of Hill [7]. For pressure loading, $\delta W$ is non-zero generally and should always be included in the type of analysis considered here.

It may be useful to establish a sufficient condition for uniqueness of the boundary value problem. Suppose that during the incremental deformation, the traction-rate is prescribed on part $S_F$ and the velocity is given on the remainder $S - S_F$ of the boundary surface. We assume that the boundary value problem has two distinct solutions and denote the difference of respective field variables of the two solutions by a prefix $\Delta$. Then

$$\Delta T^j = n_i \quad \Delta \dot{s}^{ij} = 0 \qquad \text{on } S_F$$

$$\Delta v_i = 0 \qquad\qquad \text{on } S - S_F,$$

and $\Delta \dot{s}^{ij}_{,i} = 0$ in $V$ (equilibrium condition). Hence

$$\int \Delta \dot{s}^{ij} \Delta v_{j,i} \, \mathrm{d}V - \int \Delta T^i \Delta v_i \, \mathrm{d}S = \int \Delta \dot{s}^{ij}_{,i} \Delta v_j \, \mathrm{d}V = 0.$$

Therefore, for uniqueness

$$\int \Delta \dot{s}^{ij} \Delta v_{j,i} \, \mathrm{d}V - \int n_i \Delta \dot{s}^{ij} \Delta v_j \, \mathrm{d}S > 0. \tag{6a}$$

If one of the admissible fields in (6a) is identically zero, then for a linear solid, the conditions (6) and (6a) are identical.

It may be emphasized that for pressure loading, with a given rate of increase of the pressure, $p$,

$$\Delta \dot{s}_{ij} = p \delta_{ij} \Delta v^k_{,k} - p \Delta v_{i,j},$$

and hence $\Delta \dot{s}_{ij}$ is not necessarily zero.

## MATERIAL PROPERTIES

We describe the material property of the cylinder in terms of the Jaumann derivative of the Kirchhoff stress $D\tau_{ij}/Dt$ which is related to $\dot{\sigma}_{ij}$ by (Hill [9])

$$\frac{D\tau^{ij}}{Dt} = \dot{\sigma}^{ij} + \sigma^{ij} v^k_{,k} - \sigma^{ik} \omega^j_k - \sigma^{jk} \omega^i_k. \tag{7}$$

During the incremental deformation, the constitutive equation for an isotropic elastic–plastic solid may be taken in the form (Hill [9]):

$$\begin{aligned} \frac{D\tau_{ij}}{Dt} &= 2\mu\left(\varepsilon_{ij} + \delta_{ij} \frac{\nu}{1-2\nu} \varepsilon^k_k\right) \\ &\quad - 2\mu m_{ij} \begin{cases} \dfrac{2\mu}{2\mu + h} m^k_l \varepsilon^l_k & \text{when } m_{ij} \dfrac{D\tau^{ij}}{Dt} > 0 \\ 0 & \quad \ldots \qquad\qquad \leq 0. \end{cases} \end{aligned} \tag{8}$$

The choice of the velocity field is based upon the usual assumptions made in the theory of thin shells; that is, (i) plane sections normal to the middle surface remain plane and normal; and (ii) stresses normal to the middle surface are small and could be neglected. From assumption (i) $\varepsilon_{13} = \varepsilon_{23} = 0$ and from (ii),

$$\frac{D\tau_{33}}{Dt} = 0.$$

The assumption (i) is satisfied by the following choice of the velocity field:

$$\begin{aligned} v_1 &= u - \zeta \frac{\partial w}{\partial x} \\ v_2 &= v + \frac{\zeta}{a}\left(v - \frac{\partial w}{\partial \theta}\right) \\ v_3 &= w. \end{aligned} \tag{8a}$$

The corresponding physical components of the strain-rate $e_{ij}$ are

$$e_{11} = \frac{\partial u}{\partial x} - \zeta \frac{\partial^2 w}{\partial x^2}$$

$$e_{12} = \frac{1}{2}\left[\left(\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta}\right) - \frac{\zeta}{a}\left(2\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x} + \frac{1}{a}u_\theta\right)\right] \tag{8b}$$

$$e_{22} = \frac{1}{a}\left(w + \frac{\partial v}{\partial \theta}\right) - \frac{\zeta}{a^2}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right).$$

Using assumption (ii), the constitutive equation (8) may be written in terms of physical components of the stress-rate as

$$\frac{D\tau_{11}}{Dt} = 2\mu(c_{11}e_{11} + c_{12}e_{22})$$

$$\frac{D\tau_{22}}{Dt} = 2\mu(c_{12}e_{11} + c_{22}e_{22}) \tag{9}$$

$$\frac{D\tau_{12}}{Dt} = 2\mu e_{12}.$$

where

$$c_{11} = \frac{h + 2\mu m_2^2(1+\nu)}{h(1-\nu) + 2\mu(m_1^2 + m_2^2 + 2\nu m_1 m_2)}$$

$$c_{12} = \frac{h\nu - 2\mu m_1 m_2(1+\nu)}{h(1-\upsilon) + 2\mu(m_1^2 + m_2^2 + 2\nu m_1 m_2)} \tag{10}$$

$$c_{22} = \frac{h + 2\mu m_1^2(1+\nu)}{h(1-\nu) + 2\mu(m_1^2 + m_2^2 + 2\nu m_1 m_2)}.$$

*Boundary conditions*

In order to demonstrate clearly the effect of rate of loading on the stability criterion, consider the ratio $\alpha = (\sigma_2/\sigma_1)$ and variation in this ratio. From (1)

$$2p\pi r^2 = \alpha P_z = \alpha(P + p\pi r^2)$$

and hence

$$\frac{\mathrm{d}\alpha}{\alpha}\frac{P + p\pi r^2}{P} + \frac{\mathrm{d}P}{P} - \left(\frac{\mathrm{d}p}{p} + \frac{2\mathrm{d}r}{r}\right) = 0. \tag{11}$$

Only two of the variations $\mathrm{d}\alpha$, $\mathrm{d}p$ and $\mathrm{d}P$ in (11) can be varied independently. The value of the stress at instability depends on which two of the variables are controlled, each combination in fact leads to a different boundary-value problem.

(a) Suppose that instability occurs when either (i) $p$ is held constant and $P$ is increased to a maximum, or (ii) $P$ is held constant while $p$ is increased till it reaches maximum value, or (iii) both $p$ and $P$ are increased till each reaches a maximum. In all the three cases,

$\mathrm{d}P = 0$ at instability. Swift [1] and Marciniak [2] assumed the above condition at instability. In terms of time derivative, then

$$\dot{\alpha} = 2\alpha\frac{P}{P+p\pi r^2}\frac{\dot{r}}{r},$$

and

$$\dot{s}_{11} = \frac{p}{t}r. \tag{12}$$

(b) Hillier [5] assumed the condition $\mathrm{d}p = 0$, $\mathrm{d}P_z = 0$ at instability. This assumption implies

$$\dot{\alpha} = 2\alpha(\dot{r}/r),$$

and

$$\dot{s}_{11} = 0.$$

(c) Lankford and Saibel [3] and Mellor [4] considered the case of proportional loading, for which $\mathrm{d}\alpha = 0$. These authors proposed the following conditions at instability:

(i) for $\alpha \geq \frac{1}{2}$, $\mathrm{d}p = 0$, and hence from (1) and (11),

$$\dot{s}_{11} = 2\sigma_1(\dot{r}/r), \tag{13}$$

(ii) for $\alpha \leq \frac{1}{2}$, $\mathrm{d}P = 0$. The values of $\dot{p}$ and $\dot{s}_{11}$ are then found from (1) and (11) to be

$$\dot{p} = -2p(\dot{r}/r),$$

and

$$\dot{s}_{11} = 0.$$

*Stability criterion*

It is convenient to express the stability criterion in terms of the velocity field $(u, v, w)$ at the middle surface. Hence, substituting from the equations (3), (7), (8a, b) and (9) and integrating over the thickness of the cylinder, the stability criterion (6) yields:

$$\begin{aligned}
&\int\Bigg[\left\{c_{11}u_x^2+2c_{12}u_x\frac{1}{a}(w+v_\theta)+c_{22}\frac{1}{a^2}(w+v_\theta)^2+\frac{1}{2}\left(v_x+\frac{1}{a}u_\theta\right)^2\right\}\\
&\quad+(t^2/12a^2)\Bigg\{a^2c_{11}w^2+2c_{12}(w+w_{\theta\theta})w_{xx}+c_{22}(w+w_{\theta\theta})^2\\
&\quad+\frac{1}{2}\left(2w_{x\theta}-v_x+\frac{1}{a}u_\theta\right)^2\Bigg\}-(\sigma_1/2\mu)\left\{u_x^2+\frac{1}{2}\left(v_x+\frac{1}{a}u_\theta\right)^2-v_x^2-w_x^2\right\}\\
&\quad-(\sigma_2/2\mu)\left\{\frac{1}{2}\left(v_x+\frac{1}{a}u_\theta\right)^2+\frac{1}{a^2}(w+v_\theta)^2-\frac{1}{a^2}u_\theta^2-\frac{1}{a^2}(v-w_\theta)^2\right\}\Bigg]\,\mathrm{d}S\\
&\quad-\left(\frac{1}{2}\mu t\right)\Bigg[\int\left\{pw+pwu_x-uw_x+\frac{w}{a}(w+v_\theta)+\frac{v}{a}(v-w_\theta)\right\}\mathrm{d}S_p\\
&\quad+\int n_1\dot{s}_{11}u\,\mathrm{d}S\Bigg] > 0,
\end{aligned} \tag{15}$$

where subscripts denote partial differentiation; $S_p$ is the lateral surface on which the pressure $p, \dot{p}$ act and $S$ refers to the cross sectional area.

The velocity field minimizing the stability functional is obtained from the Euler equation of (15) and may be taken in the form

$$\begin{aligned} u &= A\beta \sin\frac{m\pi x}{L}\cos n\theta \\ v &= -An\cos\frac{m\pi x}{L}\sin n\theta \\ w &= A(n^2-\beta^2 c_{11}/c_{12})\cos\frac{m\pi x}{L}\cos n\theta, \end{aligned} \tag{16}$$

where $\beta = m\pi a/L$ and $m$ and $n$ are positive integers.

Substituting $u$, $v$ and $w$ in (15) and integrating, leads to the condition which can be applied to investigate the stability of a thin cylinder under continuing deformation. In order to display clearly the role played by different loading conditions, the surface integral of the work done by the axial component of the surface traction-rate is transformed into the volume integral. The stability criterion is then found, after some manipulation, to be:

$$\begin{aligned} &\beta^4 c_{11}(c_{11}c_{22}-c_{12}^2)+\frac{t^2}{12a^2}[(n^2c_{12}-\beta^2c_{11})^2\{\beta^4c_{11}+2\beta^2(n^2-1)c_{12} \\ &+(n^2-1)^2c_{22}\}+n^2\beta^2\{(n^2-1)c_{12}-\beta^2c_{11}\}^2]-(\sigma_1/2\mu)\beta^2(\beta^2-n^2)c_{12}^2 \\ &-(\sigma_2/2\mu)[\beta^4c_{11}(2c_{11}-c_{12})-\beta^4n^2c_{11}^2+\beta^2n^2c_{12}^2+2\beta^2n^2(n^2-1) \\ &\times c_{11}c_{12}-n^4(n^2-1)c_{12}^2]-\frac{a^2c_{12}^2}{A^2\pi L\mu t}\left[\int \dot{p}w\,\mathrm{d}S_p+\int \dot{s}_{11}u_x\,\mathrm{d}V\right] > 0. \end{aligned} \tag{17}$$

## APPLICATION

(a) *Cylinder under axial compression*

In this case $p = 0$, hence $\sigma_2 = 0$. If the dead load is maintained during subsequent deformation, then $\delta W = 0$ and the stability criterion simplifies to

$$\delta E > 0.$$

For $\beta^2 \gg 1$, the value of the critical is found from (17) to be

$$\sigma_1 = 2\mu(t/a)\left[\frac{c_{11}c_{22}-c_{12}^2}{3}\right]^{\frac{1}{2}}. \tag{18}$$

A part of the material in this case may, however, be unloading and the analysis has to be modified to take unloading into account.

We have seen already that for a linear solid loading everywhere, the stability criterion and the uniqueness criterion are identical. Hence, another interpretation of (18) is possible. Suppose that during the incremental deformation, the traction-rate is presented on a part $S_F$ and the velocity is given on the remainder $S-S_F$ of the boundary surface. The value of the stress given by (18) is the condition that the boundary value problem has more than one solution. The solution for $v_i$ then consists of a homogeneous field superimposed on the non-homogeneous field (16) and, as a result, material everywhere may still be loading.

For an elastic solid, $h \to \infty$ and hence (18) reduces to

$$\sigma_1 = E(t/a)\left[\frac{1}{3(1-v^2)}\right]^{\frac{1}{2}}$$

(Timoshenko and Gere [10], equation 11-1).

(b) *Cylinder under external pressure* $p$ ($\dot{p} = 0$)

We assume the cylinder to deform in a state of plane strain, deformation parallel to the axes being prevented. The value of the critical pressure $p$, in this case, is obtained by putting $\beta = 0$ in (17); thus

$$p < (n^2-1)\frac{t^2}{12a^2}2\mu c_{22}.$$

$$p_{\text{crit}} \equiv \frac{t^2}{4a^2}2\mu c_{22}.$$

For an elastic solid, $c_{22} = 1/(1-v)$, and the value of $p_{\text{crit}}$ at onset of instability then coincides with the classical result. (Timoshenko and Gere [10], equation 7.15).

(c) *Axi-symmetric deformation*

The stability criterion for axi-symmetric deformation is obtained by putting $n = 0$ in equation (17),

$$\begin{aligned} &c_{11}(c_{11}c_{22}-c_{12}^2)+\left(\frac{t^2}{12a^2}\right)c_{11}^2(\beta^4c_{11}-2\beta^2c_{12}+c_{22}) \\ &-(\sigma_1/2\mu)c_{12}^2-(\sigma_2/2\mu)c_{11}(2c_{11}-c_{12}) \\ &-\frac{a^2c_{12}^2}{A^2\pi\mu Lt}\left[\int \dot{p}w\,\mathrm{d}S_p+\int \dot{s}_{11}u_x\,\mathrm{d}V\right] > 0. \end{aligned} \tag{19}$$

The term containing $(t^2/12a^2)$ is due to shear stiffening and is neglected in the following analysis.

(i) Consider the condition for instability proposed by Swift [1] and by Marciniak [2], namely $\mathrm{d}p = 0$, $\mathrm{d}P = 0$; consequently $\dot{s}_{11} = (p/t)w$. Substituting in (17) and using (10), we obtain, at instability:

$$\sigma_1 = \frac{\dfrac{1}{E}+\dfrac{m_2^2}{h}}{\left(\dfrac{v}{E}-\dfrac{m_1m_2}{h}\right)^2+2(\sigma_2/\sigma_1)\left(\dfrac{1}{E}+\dfrac{m_2^2}{h}\right)\left(\dfrac{1-v}{E}-\dfrac{m_2m_3}{h}\right)}. \tag{20}$$

In deriving (20), no assumption has been made regarding the shape of the yield surface. Hence the result is applicable to a wide class of inelastic solids. For a solid obeying von Mises yield criterion,

$$m_1 = \frac{2-\alpha}{\sqrt{[6(1-\alpha+\alpha^2)]}}, \qquad m_2 = \frac{2\alpha-1}{\sqrt{[6(1-\alpha+\alpha^2)]}}, \qquad m_3 = -\frac{1+\alpha}{\sqrt{[6(1-\alpha+\alpha^2)]}}. \tag{21}$$

The value of $m_{ij}$ for Tresca solid will depend on the value of $\alpha$. For $0 < \alpha < 1$, $m_1 = 1/\sqrt{2}, m_2 = 0, m_3 = -1/\sqrt{2}$; and for $1 < \alpha < \infty, m_1 = 0, m_2 = 1/\sqrt{2}, m_3 = -1/\sqrt{2}$. However, for $\alpha = 1$, Tresca yield surface (Fig. 1) has a fan of normals lying between Normal A and Normal C.

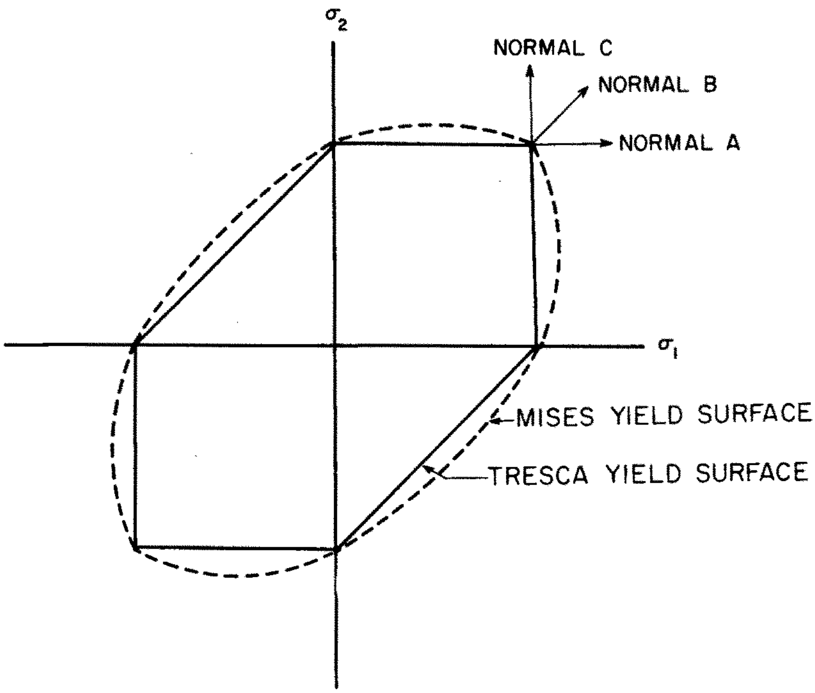


FIG. 1.

Equation (20) is in a very convenient form to examine the sensitivity of the critical stress to the variation in normal $m_{ij}$. In Fig. 2, the value of $(\sigma_1/E)$ is shown against $\lambda(E/E_t)$ for $\alpha = 1$ and corresponding to Normal $B$ and Normal $C$. It is seen that the critical stress corresponding to the Normal $C$ of Tresca solid is significantly lower than the critical stress corresponding to the Normal $B$ of von Mises solid. The critical stress is found to be quite sensitive to the variation in $m_{ij}$ for other values of $\alpha$ as well. This is a significant observation and is in agreement with the observation of Sewell [11] and Ariaratnam and Dubey [12].

It is seen, further, that the value of the critical stress decreases very rapidly as the value of $\lambda$ increases. For sufficiently large $\lambda$, the critical stress is found to be of the same order of magnitude as that of the work-hardening parameter $h$.

For a rigid–plastic solid $E \to \infty$ and $E_t = \frac{3}{2}h$ (von Mises solid). Using this relation, equations (21) and the relation

$$\bar{\sigma}^2 = \tfrac{3}{2}\sigma'_{ij}\sigma'_{ij} = (\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2), \tag{22}$$

in (20), we obtain

$$\frac{1}{\bar{\sigma}}E_t = \frac{4 - 6\alpha + 3\alpha^2 + 4\alpha^3}{4(1 - \alpha + \alpha^2)^{\frac{3}{2}}}.$$

This result has been obtained by Swift and by Marciniak using different methods.

(ii) Using the condition $dp = 0$, $dP_z = 0$, hence $\dot{s}_{11} = 0$, we obtain at instability:

$$\sigma_1 = \frac{\dfrac{1}{E}+\dfrac{m_2^2}{h}}{\left(\dfrac{v}{E}-\dfrac{m_1m_2}{h}\right)^2+\alpha\left(\dfrac{1}{E}+\dfrac{m_2^2}{h}\right)\left[\dfrac{2-v}{E}+\dfrac{m_2(m_1+2m_2)}{h}\right]}. \tag{23}$$

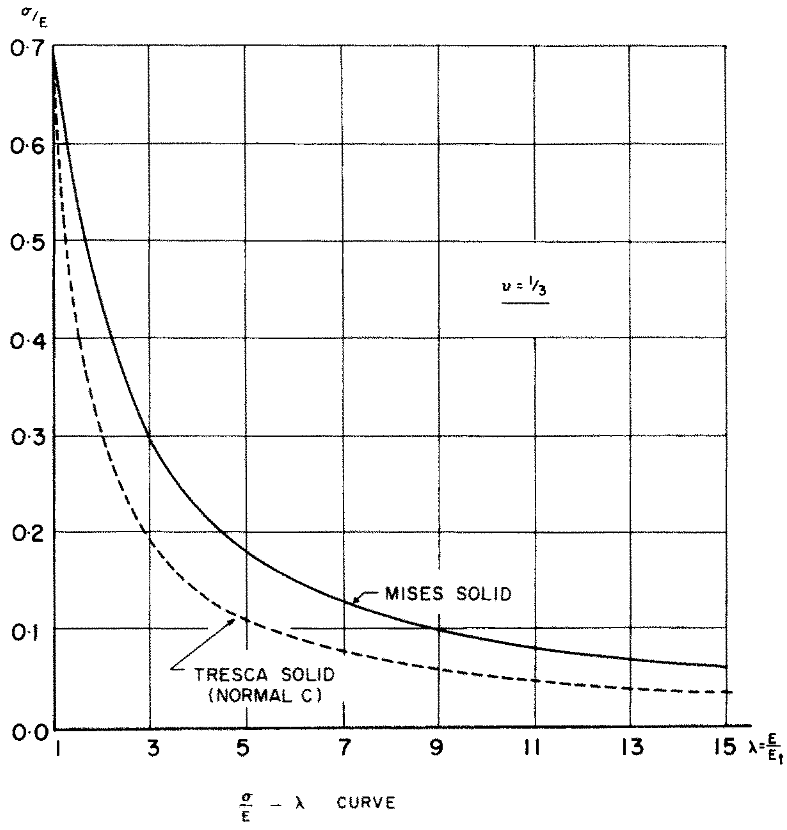


Fig. 2.

For a rigid–plastic solid obeying von Mises yield criterion, (23) reduces to the following condition for instability:

$$\frac{1}{\bar{\sigma}}E_t = \frac{2-2\alpha-\alpha^2+3\alpha^3}{2(1-\alpha+\alpha^2)^{\frac{3}{2}}},$$

which was first obtained by Hillier [5].

(iii) We assume now that the deformation continues under proportional stressing, so that $d\alpha = 0$. Assuming, further, that for $\alpha \geq \frac{1}{2}$, the instability occurs when $dp = 0$

and hence $\dot{s}_{11} = 2\sigma_1(w/a)$. Substituting in (19), and using (10), it follows that for instability:

$$\sigma_1 = \frac{\dfrac{1}{E}+\dfrac{m_2^2}{h}}{\left[\dfrac{2-\nu}{E}+\dfrac{m_2(m_1+2m_2)}{h}\right]\left[\alpha\left(\dfrac{1}{E}+\dfrac{m_2^2}{h}\right)-\dfrac{\nu}{E}+\dfrac{m_1 m_2}{h}\right]}. \tag{24}$$

For a von Mises solid having infinitely large modulus of elasticity, (24) yields

$$\frac{1}{\bar{\sigma}}E_t = \frac{3\alpha}{2(1-\alpha+\alpha^2)^{\frac{1}{2}}}.$$

This is the same as the result obtained by Lankford and Saibel and by Mellor.

These authors proposed that for $\alpha \leq \frac{1}{2}$, $dP = 0$, hence $\dot{s}_{11} = 0$ and $\dot{p} = -2p(w/a)$ at instability. Hence from (17), the critical stress at instability is found to be

$$\sigma_1 = \frac{\dfrac{1}{E}+\dfrac{m_2^2}{h}}{\left(\dfrac{\nu}{E}-\dfrac{m_1 m_2}{h}\right)^2-\alpha\left(\dfrac{1}{E}+\dfrac{m_2^2}{h}\right)\left(\dfrac{\nu}{E}-\dfrac{m_1 m_2}{h}\right)}. \tag{25}$$

From (25), we obtain the condition

$$\frac{1}{\bar{\sigma}}E_t = \frac{2-\alpha}{2(1-\alpha+\alpha^2)^{\frac{1}{2}}},$$

at instability for a rigid–plastic (von Mises) solid.

(iv) Cooper's result, as generalized by Hillier, follows by neglecting the contribution due to $\delta W$. The condition for instability for a rigid–plastic solid, then emerges in the form:

$$\frac{1}{\bar{\sigma}}E_t = \frac{(1+\alpha)(4-7\alpha+4\alpha^2)}{4(1-\alpha+\alpha^2)^{\frac{3}{2}}}.$$

Thus, from equations (20), (23–25), we obtain the value of the critical stress, at instability, for a cylinder under internal pressure $p$ and independent axial load $P$, generalized for an elastic–plastic solid. The difference in the result is due to different loading conditions imposed. Another feature of the analysis is the fact that the results obtained here can be applied to a wide class of inelastic solids, von Mises solid and Tresca solid being included as special cases.

The value of the critical stress at instability is found to be quite sensitive to the variation in the direction of the unit normal of the yield surface at local stress point. This is one of the features of the present analysis. The variation in the normal could be due to the stress ratio being different than its ideal value or due to different material behaviour from the idealized case. Further, the critical stress is found to decrease very rapidly as the value of $\lambda(=E/E_t)$ increases, thus bringing the result obtained here in the range of possible practical interest. The result is expected to be useful specially for analysis in pressure vessels, where elastic deformation may not be negligible.

For the boundary value problem of the type considered, different boundary conditions are admissible, each leading to different results. In fact, the results of the authors [1–5] follow as special cases under different boundary conditions imposed.

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## APPENDIX

The main structural problem is to calculate the value of the maximum load that the system can stand. For $\alpha(=\sigma_2/\sigma_1)$, it is seen from (1) that

$$P_z = \sigma_1(2\pi rt) \quad \text{and} \quad p = \sigma_1(t/r).$$

Hence the degree of sensitivity of the stress to the variation in the normal to the yield surface is reflected in the sensitivity of the load in the same proportion. A numerical prediction can be made only if the stress–strain relationship for the material is known precisely. To the author's knowledge, a general stress–strain law, covering the entire range of deformation starting with purely elastic deformation to predominantly plastic deformation, is unavailable and this makes the question of predicting the variation in the critical stress an awkward one to answer. However, if the deformation is assumed to be predominantly plastic then we can use the relation (Johnson and Mellor [14])

$$\sigma = 22{,}200\,\varepsilon^{0\cdot 25} \tag{26}$$

which holds for soft aluminum. With the help of (8) and (26), the values of the critical stress can be obtained from (20) or from (23) and are given below:

(a) Using Swift's criterion (20),

$$\sigma_1 = \begin{cases} 14848 \text{ psi corresponding to Normal } B \text{ of von Mises solid} \\ 13197 \text{ psi corresponding to Normal } C \text{ of Tresca solid;} \end{cases}$$

(b) Using Hillier's criterion (23),

$$\sigma_1 = \begin{cases} 15695 \text{ psi corresponding to Normal } B. \\ 13197 \text{ psi corresponding to Normal } C. \end{cases}$$

However, it is felt that further verification is necessary before the usefulness of the theory can be established.

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**Абстракт**—Исследуется условие неустойчивости изотропного, упругопластического цилиндра, подверженного действию давления р и независимой осевой нагрузки Р. Оказывается, что граничная задача доиускает разные условия наярузки, лричем каждое условие приводит к разному результату. В качестве специалъных случаев даются критические напряжения для цилиндров нагруженных вдолъ оси и цилиндров под влиянием одномерно горизонталъного давления. Далее, локазано, что значение критического напряжения является оченъ чувствителъным на изменение нормали к поверхности, в местной точке напряжения.